ON THE INDEPENDENCE OF A GENERALIZED STATEMENT OF EGOROFF'S THEOREM FROM ZFC, AFTER T. WEISS

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ABSTRACT. We consider a generalized version (GES) of the well-known Severini–Egoroff theorem in real analysis, first shown to be undecidable in ZFC by Tomasz Weiss in [4]. This independence is easily derived from suitable hypotheses on some cardinal characteristics of the continuum like $\mathfrak{b}$ and $\mathfrak{o}$, the latter being the least cardinality of a subset of $[0,1]$ having full outer measure.

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In this paper we will consider the following *Generalized Egoroff Statement*, which is a version "without regularity assumptions" of the well-known Severini–Egoroff theorem from real analysis:

GES Given a sequence $(f_n : n \in \mathbb{N})$ of arbitrary functions $[0,1] \to \mathbb{R}$ converging pointwise to 0, for each $\eta > 0$ there is a subset $A \subseteq [0,1]$ of outer measure $\mu^*(A) > 1 - \eta$ such that (f_n) converges uniformly on A.

This conjecture first emerged from some questions about the behaviour of bounded harmonic functions on the unit disc in $\mathbb{C}$; in particular, it has been used in [2] to show the independence from ZFC of a strong Littlewood-type statement about tangential approaches.

Notice that in GES it is necessary to consider Lebesgue *outer* measure to avoid simple counterexamples in ZFC:

Proposition 1. *There is a decreasing sequence $(f_n : n \in \mathbb{N})$ of functions $[0,1] \to \mathbb{R}$, converging pointwise to zero, such that every subset $A \subseteq [0,1]$ on which (f_n) converges uniformly has Lebesgue inner measure zero.*

Proof. By a theorem of Lusin and Sierpiński there exists a partition of $[0,1]$ into countably many (in fact, even continuum many) pieces

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$\{B_n : n \in \mathbb{N}\}$ each having full outer measure. Consider then the sequence (f_n) where, for every $n \in \mathbb{N}$, f_n is the characteristic function of the subset $B_{\geq n} = \bigcup_{k \geq n} B_k$ of the unit interval: clearly $(f_n(x))$ converges monotonically to zero on every point $x \in [0,1]$; if (f_n) converges uniformly on a subset A, A has to be disjoint from $B_{\geq \bar{n}}$ for some $\bar{n} \in \mathbb{N}$, so $\mu_*(A) \leq 1 - \mu^*(B_{\geq \bar{n}}) = 0$. $\square$

Fix once and for all a decreasing vanishing sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers, e.g. $\varepsilon_n = 2^{-n}$; consider the following function, mapping a sequence of reals to its (ε-)*order of convergence* to zero:

$$\begin{aligned} &\mathrm{oc} : c_0 \to {}^{\mathbb{N}}\mathbb{N}{\uparrow}, \quad \text{defined on each } a = (a_n) \in c_0 \text{ as} \\ &(\mathrm{oc}\, a)_n = \min\left\{m : \forall l \geq m \left(|a_l| \leq \varepsilon_n\right)\right\}, \end{aligned} \tag{1}$$

where c_0 denotes the set of infinitesimal real-valued sequences and ${}^{\mathbb{N}}\mathbb{N}{\uparrow} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is the set of nondecreasing sequences of natural numbers.

Using the natural identification of ${}^{\mathbb{N}}({}^{X}\mathbb{R})$ with ${}^{X}({}^{\mathbb{N}}\mathbb{R})$, we can view a sequence of real-valued functions $X \to \mathbb{R}$ converging pointwise to zero as a single function $F : X \to c_0$, and then study the associated order of convergence, $\mathrm{oc}\, F = \mathrm{oc} \circ F : X \to {}^{\mathbb{N}}\mathbb{N}{\uparrow}$:

Lemma 2. *F converges uniformly to zero if and only if the range of $\mathrm{oc}\, F$ is bounded in $({}^{\mathbb{N}}\mathbb{N}, \leq)$, where $\leq$ is the partial order of everywhere domination: $\alpha \leq \beta$ iff $\forall n\ (\alpha_n \leq \beta_n)$.*

Proof. This is just a restatement of the definition of uniform convergence:

$$\begin{aligned} &F \text{ converges uniformly to } 0 \quad \leftrightarrow \\ \leftrightarrow \quad &\forall n\, \exists m\, \forall x \in X\, \forall l \geq m \left(|F_l(x)| \leq \varepsilon_n\right) \quad \leftrightarrow \\ \leftrightarrow \quad &\exists\, (m_n) \in {}^{\mathbb{N}}\mathbb{N}\ \forall n\, \forall x \in X \left((\mathrm{oc}\, F(x))_n \leq m_n\right). \end{aligned}$$

$\square$

Lemma 3. *For all $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}{\uparrow}$ there exists a sequence F of real-valued functions on X converging pointwise to 0 with order $\mathrm{oc}\, F = \varphi$.*

Proof. It is sufficient to prove the lemma pointwise: given a nondecreasing sequence of natural numbers $\alpha \in {}^{\mathbb{N}}\mathbb{N}{\uparrow}$, we construct a sequence $a \in c_0$ converging to 0 with order α. For that, just let

$$a = (a_n)_{n \in \mathbb{N}} \quad \text{where} \quad a_n = \inf\{\varepsilon_k : \alpha_k \leq n\};$$

it is straightforward to check that this works, i.e. $\mathrm{oc}\, a = \alpha$. $\square$

Let μ^* be an upward continuous outer measure on a set X, i.e. an outer measure $\mathrm{Pow}\, X \to [0, +\infty]$ satisfying

$$A = \bigcup_{n \in \mathbb{N}} A_n \quad \to \quad \mu^*(A) = \lim_{n \to \infty} \mu^*\Big(\bigcup_{k<n} A_k\Big).$$

For every sequence F of real-valued functions on X converging pointwise to zero, consider the statement

$\mathsf{GES}(X,\mu^*,F)$ for each $M < \mu^*(X)$ there is a subset $A \subseteq X$ such that $\mu^*(A) > M$ and F converges uniformly on A;

the Generalized Egoroff Statement relative to the space (X,μ^*) is the formula

$$\mathsf{GES}(X,\mu^*) = \forall F\ \mathsf{GES}(X,\mu^*,F);$$

clearly our original statement GES is just $\mathsf{GES}([0,1],m^*)$, where m^* is Lebesgue outer measure on the unit interval $[0,1] \subseteq \mathbb{R}$. Denote by $\mathcal{K}_\sigma$ the σ-ideal generated by the bounded subsets of $({}^{\mathbb{N}}\mathbb{N},\leq)$; equivalently, $\mathcal{K}_\sigma$ is the family of those subsets which are bounded with respect to the order $\leq^*$ of eventual domination,

$$\alpha \leq^* \beta \ \leftrightarrow\ \forall^\infty n\ (\alpha_n \leq \beta_n)\ \leftrightarrow\ \exists n\ \forall k \geq n\ (\alpha_k \leq \beta_k) \quad (\alpha,\beta \in {}^{\mathbb{N}}\mathbb{N}),$$

and $\mathcal{K}_\sigma$ is also the σ-ideal generated by the compact subsets of the Baire space ${}^{\mathbb{N}}\mathbb{N}$ (see [3]).

Lemma 4. $\mathsf{GES}(X,\mu^*,F)$ *holds iff there is a subset* $Y \subseteq X$ *of full outer measure (i.e.* $\mu^*(Y) = \mu^*(X)$*) such that* $\operatorname{oc} F[Y] \in \mathcal{K}_\sigma$.

Proof. Fix an increasing sequence of positive real numbers (M_n) with limit $\mu^*(X)$. Assume $\mathsf{GES}(X,\mu^*,F)$: by lemma 2, for every $n \in \mathbb{N}$ there is a subset $A_n \subseteq X$ such that $\mu^*(A_n) > M_n$ and $\operatorname{oc} F[A_n]$ is bounded in ${}^{\mathbb{N}}\mathbb{N}$; taking $Y = \bigcup_{n\in\mathbb{N}} A_n$, Y has full outer measure and $\operatorname{oc} F[Y] = \bigcup_{n\in\mathbb{N}} \operatorname{oc} F[A_n]$ is σ-bounded, as required. Conversely, suppose that $\mu^*(Y) = \mu^*(X)$ and $\operatorname{oc} F[Y] \subseteq \bigcup_{n\in\mathbb{N}} B_n$, where each B_n is a bounded subset of $({}^{\mathbb{N}}\mathbb{N},\leq)$, and put

$$A_n = (\operatorname{oc} F)^{-1}[B_0 \cup \ldots \cup B_{n-1}]:$$

since $\operatorname{oc} F[A_n]$ is bounded, F converges uniformly on every A_n (lemma 2); moreover, as μ^* is continuous and $Y \subseteq \bigcup_{n\in\mathbb{N}} A_n$, for all m there is some n such that $\mu^*(A_n) > M_m$, that is, $\mathsf{GES}(X,\mu^*,F)$ holds. $\square$

Theorem 5. $\mathsf{GES}(X,\mu^*)$ *holds if and only if for all functions* $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}$ *there is a subset* $Y \subseteq X$ *of full outer measure such that* $\varphi[Y] \in \mathcal{K}_\sigma$.

This theorem provides a translation of GES into a purely set-theoretical statement.

Proof. The "if" direction follows directly from lemma 4 using $\varphi = \operatorname{oc} F$. For the converse, consider the function Θ which maps a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ to the nondecreasing sequence $\left(\sum_{k\leq n} \alpha_k\right)_{n\in\mathbb{N}}$: it is a bijective order morphism $({}^{\mathbb{N}}\mathbb{N},\leq) \to ({}^{\mathbb{N}}\mathbb{N}{\uparrow},\leq)$ satisfying $\alpha \leq \Theta(\alpha)$, therefore, for all $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$, $\Theta[Y]$ is (σ-)bounded iff Y is (σ-)bounded. Assume $\mathsf{GES}(X,\mu^*)$ and let φ be a function $X \to {}^{\mathbb{N}}\mathbb{N}$; by lemma 3 there exists a sequence F of real-valued functions converging pointwise to 0 with $\operatorname{oc} F = \Theta \circ \varphi$, so there is a set $Y \subseteq X$ of full outer measure such that $\Theta[\varphi[Y]] = \operatorname{oc} F[Y] \in \mathcal{K}_\sigma$ (lemma 4), i.e. $\varphi[Y] \in \mathcal{K}_\sigma$ as desired. $\square$

Remark. Theorem 5 is still valid for measure spaces (X, μ) and the classical Egoroff Statement, provided that we only consider measurable maps φ and measurable subsets $Y \subseteq X$. Thus theorem 5 entails the Severini–Egoroff theorem: if μ is finite and $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}$ is measurable, the image measure $\varphi_* \mu$ is a finite Borel measure on ${}^{\mathbb{N}}\mathbb{N}$, hence it is regular and it is always supported by a σ-compact subset.

Recall that the *bounding number* $\mathfrak{b} = \mathrm{non}(\mathcal{K}_\sigma)$ (see [3]) is the smallest possible size of a subset of ${}^{\mathbb{N}}\mathbb{N}$ not belonging to $\mathcal{K}_\sigma$. We also denote with $\mathfrak{o}(X, \mu^*)$ the least cardinality of a subset of X having full outer measure and let $\mathfrak{o} = \mathfrak{o}([0,1], m^*)$[1].

Corollary 6. *Assuming* $\mathfrak{o}(X, \mu^*) < \mathfrak{b}$, $\mathsf{GES}(X, \mu^*)$ *holds. In particular,* $\mathfrak{o} < \mathfrak{b}$ *implies* GES[2].

Proof. Fix a subset $Y \subseteq X$ of full outer measure with $|Y| = \mathfrak{o}(X, \mu^*)$; then every function $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}$ maps Y onto a set of cardinality less than $\mathfrak{b}$, hence $\varphi[Y] \in \mathcal{K}_\sigma$. □

We can also invoke theorem 5 to prove sufficient conditions for the failure of GES. Precisely, we infer $\neg\mathsf{GES}(X, \mu^*)$ by constructing (under suitable hypotheses) a set $Z \subseteq {}^{\mathbb{N}}\mathbb{N}$ of cardinality $|Z| \geq |X|$ such that all subsets of Z belonging to $\mathcal{K}_\sigma$ have size less than $\mathfrak{o}(X, \mu^*)$: once this is achieved, if φ is any injection $X \to Z$, no subset $Y \subseteq X$ of full measure can be mapped onto an element of $\mathcal{K}_\sigma$, because $|\varphi[Y]| = |Y| \geq \mathfrak{o}(X, \mu^*)$. In order to state the next proposition, we recall that the *dominating number* $\mathfrak{d} \geq \mathfrak{b}$ is the least cardinality of a cofinal subset of $({}^{\mathbb{N}}\mathbb{N}, \leq^*)$ and that a *κ-Lusin set* is a subset $L \subseteq \mathbb{R}$ of cardinality κ whose meager (i.e. Baire first category) subsets have size less than κ.

Proposition 7. *Assume* $\mathfrak{o}(X, \mu^*) = |X| = \kappa$*; then* $\mathsf{GES}(X, \mu^*)$ *fails in each of the following cases:*

1. $\kappa = \mathfrak{b}$*;*
2. $\kappa = \mathfrak{d}$*;*
3. *there exists a κ-Lusin set.*

Proof. Following the plan outlined before stating the proposition, we try to build a "κ-Lusin set" Z for the ideal $\mathcal{K}_\sigma$ instead of the ideal of meager sets. This is automatic under hypothesis (3): every (true) κ-Lusin set has the required properties, since all compact subsets of ${}^{\mathbb{N}}\mathbb{N}$ have empty interior and thus every $\mathcal{K}_\sigma$ set is meager.

Assume $\kappa = \mathfrak{b}$ and let $\{\alpha^\xi\}_{\xi<\mathfrak{b}}$ be an unbounded family in $({}^{\mathbb{N}}\mathbb{N}, \leq^*)$. By transfinite recursion we build a wellordered unbounded chain $Z = \{\beta^\xi\}_{\xi<\mathfrak{b}}$ of length $\mathfrak{b}$: after the construction of all β^η for $\eta < \xi$, pick β^ξ

[1] We haven't been able to find any specific name for this cardinal in the literature.
[2] The latter fact has been pointed out by T. Weiss and I. Recław (see [4]).

among the strict $\leq^*$-upper bounds of the set $\left\{\alpha^\xi\right\} \cup \{\beta^\eta\}_{\eta<\xi}$ (which has size less than $\mathfrak{b}$ and thus is $\leq^*$-bounded). It is clear that no $\leq^*$-bounded subset of Z can be cofinal in Z, hence all $\mathcal{K}_\sigma$ subsets of Z have cardinality $< \mathfrak{b}$.

Finally, suppose $\kappa = \mathfrak{d}$ and let $\left\{\alpha^\xi\right\}_{\xi<\mathfrak{d}}$ be a cofinal family in $({}^{\mathbb{N}}\mathbb{N}, \leq^*)$. We build a set $Z = \left\{\beta^\xi\right\}_{\xi<\mathfrak{d}}$ of cardinality $\mathfrak{d}$ by transfinite recursion as follows: after the construction of all β^η for $\eta < \xi$, pick an element β^ξ which is not $\leq^*$ any element of the set $\{\alpha^\eta\}_{\eta\leq\xi} \cup \{\beta^\eta\}_{\eta<\xi}$ (which has size less than $\mathfrak{d}$ and thus is not $\leq^*$-cofinal). Z has the desired properties: $\left(\beta^\xi\right)_{\xi<\mathfrak{d}}$ is a sequence without repetitions, hence $|Z| = \mathfrak{d}$, and moreover, if $A \subseteq Z$ is in $\mathcal{K}_\sigma$, some α^ξ has to eventually dominate all elements of A, which implies that $A \subseteq \{\beta^\eta\}_{\eta<\xi}$ has cardinality less than $\mathfrak{d}$. $\square$

Corollary 8. GES *fails whenever at least one of the following hypotheses is satisfied:*

(1) $\mathfrak{o} = \mathfrak{d} = \mathfrak{c}$ *(the cardinality of the continuum);*
(2) *there exists a* $\mathfrak{c}$*-Lusin set and* $\mathfrak{o} = \mathfrak{c}$*;*
(3) *there exists a* $\mathfrak{c}$*-Lusin set and* $\mathfrak{c}$ *is a regular cardinal.*

The last two conditions provide an affirmative answer (at least when $\mathfrak{c}$ is regular or it coincides with $\mathfrak{o}$) to a question posed by T. Weiss; he also noticed that there are models of ZFC (e.g. the iterated Mathias real model, where $\mathfrak{o} = \mathfrak{d} = \mathfrak{c}$) which contain no $\mathfrak{c}$-Lusin sets but nevertheless satisfy $\neg$GES.

Proof. Assumptions (1) and (2) are just particular instances of cases (2) and (3) respectively of proposition 7. Moreover, hypothesis (3) is stronger than both (1) and (2): if κ is a regular cardinal and there is a κ-Lusin set, then $\mathrm{cov}(\mathcal{M}) \geq \kappa$ and thus $\mathfrak{d} \geq \mathrm{cov}(\mathcal{M}) \geq \kappa$ and $\mathfrak{o} \geq \mathrm{non}(\mathcal{N}) \geq \mathrm{cov}(\mathcal{M}) \geq \kappa$ (see [1] for the relevant definitions of these cardinal characteristics associated to the σ-ideals $\mathcal{M}$ of meager sets and $\mathcal{N}$ of Lebesgue nullsets, as well as for the proofs in ZFC of the stated inequalities). $\square$

Corollary 9 (T. Weiss)**.** GES *is undecidable in* ZFC*.*

Proof. The hypothesis of corollary 6, and therefore GES, hold in the iterated Laver real model (see [1] and the proof of theorem 1 in [4]). On the other hand, $\mathfrak{o} = \mathfrak{d} = \mathfrak{c}$ is certainly true (thus $\neg$GES holds) under the Continuum Hypothesis CH or just Martin's Axiom MA, which are consistent with ZFC. $\square$

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